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# Engineering Notes

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## Input-to-State Stable Attitude Control

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### I. Introduction

It has been recently shown in [1,2] that there exist stabilizing proportional-plus-derivative (PD)-type control laws for the problem of attitude stabilization under the so-called Rodriguez or modified Rodriguez orientation parameters [3]. The contribution in [1,2] depends on passivity theory and the author shows that there exists a storage function that describes the passivity of the system.

In this Note, we show via simulation that these PD-type control laws are not generally robust with respect to bounded disturbances and provide a sufficient condition under which nonlinear-type control laws can render the system semiglobally input-to-state stable. As such, the closed-loop system is robust with respect to any disturbance within a quantifiable restriction on the amplitude, as well as the set of initial conditions, if the control gains are designed appropriately.

The rest of this Note is organized as follows: In Sec. II, we state the underlying kinematic model and provide an example on the lack of robustness in passivity-based control laws proposed in [1]. In Sec. III, we review the basic definitions of input-to-state stability (ISS) and show that our proposed nonlinear control law guarantees semiglobal ISS with respect to input disturbances. We show in Sec. IV that the same example, which is provided under the proposed control law, is stable under identical conditions.

### II. Problem Statement

Let us consider the following full kinematic model (see [1–3] for more details):

$$\dot{\rho} = H(\rho)w, \qquad J\dot{w} = S(w)Jw + u \tag{1}$$

where $\rho \in \mathbf{R}^3$ is the so-called Cayley–Rodrigues parametrization, $w \in \mathbf{R}^3$ is the angular velocity vector, $J$ is the inertia matrix,

$$S(x) = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}$$

for any $x \in \mathbf{R}^3$ is a skew-symmetric matrix with the property $S(x)y = -S(y)x$ for any two vectors $x, y \in \mathbf{R}^3$, and $H(\rho) = \frac{1}{2}(I - S(\rho) + \rho\rho^T)$. It is easy to show that the following property holds [1,2]: $\rho^T H(\rho) = \frac{1}{2}(1 + \rho^T\rho)\rho^T$.

In this Note we propose using the following nonlinear control input:

$$u = -k_1 w - k_2(1 + \rho^T\rho)\rho \tag{2}$$

where $k_1, k_2 > 0$ are some scalar gains for simplicity,‡ which will be shown to be semiglobally input-to-state stable, as opposed to the simple PD-type control law given by

$$u = -k_1 w - k_2 \rho \tag{3}$$

As a motivating example, let us consider the problem of controlling system (1) with the PD-type control law (3). Let us assume that the closed-loop systems (1) and (3) are acted upon by a bounded disturbance $d$; that is, the input in Eq. (3) is given by $u = -k_1 w - k_2 \rho + d$. Let us take this extra disturbance input to be the following (this example was inspired by the one in [4]):

$$d = 100\,\mathrm{sgn}(w)$$

where

$$\mathrm{sgn}\,(w) = \begin{cases} 1, & \text{if } w > 0 \\ 0, & \text{if } w = 0 \\ -1, & \text{if } w < 0 \end{cases}$$

The simulation in Fig. 1 shows the response of the system with $k_1 = k_2 = 10$ and $J = I_{3\times3}$ up to 0.25 s, after which the solution of the system diverges very fast.

Therefore, a bounded disturbance input can actually destabilize the system quite fast, and the need arises to characterize how robust a specific control law is with respect to bounded disturbances.

### III. Main Result

The concept of ISS is well suited to characterize the stability of the system under bounded disturbances (see [5] and references therein). Before we state our main result, let us first state the following definition of ISS.

*Definition*. The closed-loop system (1) and (2) is semiglobally ISS with respect to state $x = [\rho^T, w^T]^T$ and input $d$ if for any positive constants $\Delta_x$ and $\Delta_d$ there exist gains $k_1$ and $k_2$ such that given $\|x_0\| < \Delta_x$ and $\sup_{t\geq 0}\|d(t)\| < \Delta_d$, the solution $x(t)$ is defined $\forall\ t > 0$ and the following bound holds on the state

$$\|x(t)\| \leq \max\left\{\beta(\|x_0\|, t), \gamma\left(\sup_{0\leq\tau\leq t}\|d(\tau)\|\right)\right\}$$

for some functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$.§

We are ready to state the main result of this Note.

*Theorem*. Given any initial condition $x_0 = x(0)$, there exist large enough control gains $k_1$ and $k_2$ that render the closed-loop system (1) and (2) semiglobally ISS from an external disturbance input $d$ to the

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‡We can also take matrix gains such that $K_i = K_i^T > 0$ $(i \in \{1, 2\})$.

§A continuous function $\alpha$:$[0, a) \to [0, \infty)$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\beta$:$[0, a) \times [0, \infty) \to [0, \infty)$ belongs to class $\mathcal{KL}$ if for each fixed $s$ the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to $r$ and for each fixed $r$ the mapping $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \to 0$ as $s \to \infty$ [6].

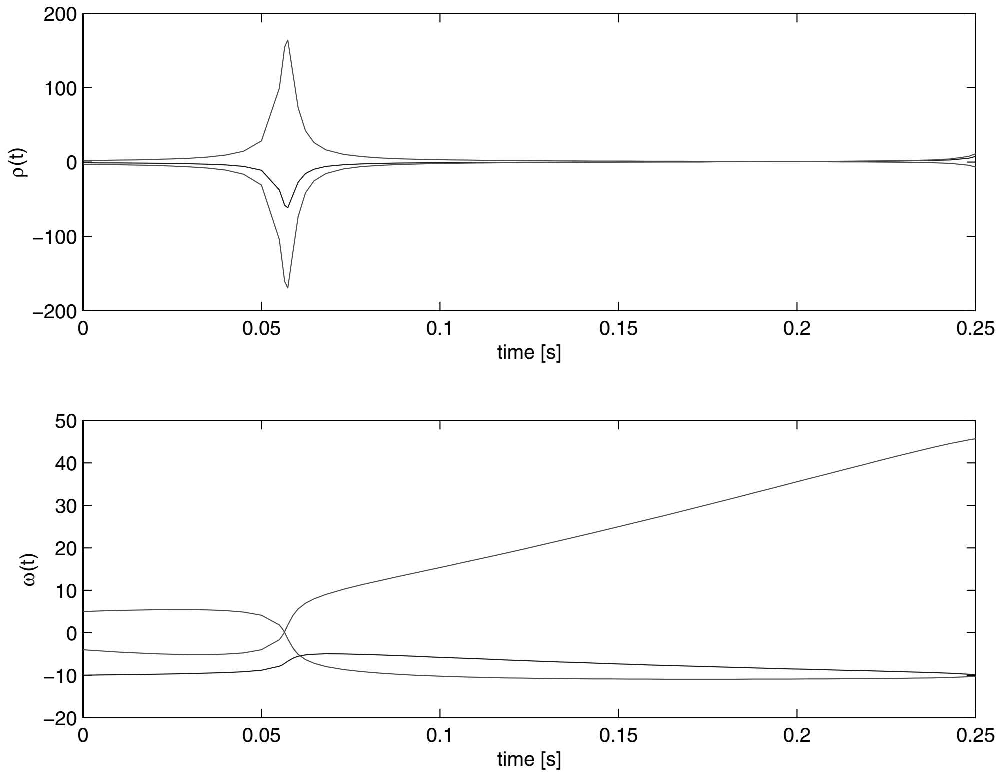


**Fig. 1 Instability of passivity-based control laws under bounded disturbances.**

state $x = [w^T, \rho^T]^T$ with some restrictions $\Delta_x$ and $\Delta_d$ on the state and disturbance, respectively.

*Proof*. Consider

$$V(x) = V(w, \rho) = \tfrac{1}{2} w^T J w + k_2 \rho^T \rho + \epsilon \rho^T J w$$

First, we can easily show that the following quadratic bounds hold on $V$:

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2 \tag{4}$$

where

$$c_1 < \min\left\{\frac{\lambda_{\min}(J)}{2}, k_2\right\}$$

and

$$c_2 > \max\left\{\frac{\lambda_{\max}(J)}{2}, k_2\right\}$$

are positive constants as long as

$$\epsilon < \frac{2}{\lambda_{\max}(J)} \times \min\left\{\sqrt{(c_2 - k_2)\left(c_2 - \frac{\lambda_{\max}(J)}{2}\right)}, \sqrt{(k_2 - c_1)\left(\frac{\lambda_{\min}(J)}{2} - c_1\right)}\right\}$$

where $\lambda_{\min}(J)$ and $\lambda_{\max}(J)$ are the smallest and largest eigenvalues of $J$, respectively. Upon taking the derivative of $V$ along the trajectories of Eqs. (1) and (2), we obtain that

$$\dot{V}(x) = -k_1 w^T w - \epsilon k_2 (1 + \rho^T \rho) \rho^T \rho + \epsilon \rho^T S(w) J w - \epsilon k_1 \rho^T w + \epsilon w^T J^T H(\rho) w + w^T d + \epsilon \rho^T d$$

Because $S(w)w = 0$ and $\rho^T H(\rho) = \frac{1}{2}(1 + \rho^T \rho)\rho^T$, we can write $\dot{V}(x)$ as

$$\dot{V}(x) = -k_1 w^T w - \epsilon k_2 (1 + \rho^T \rho) \rho^T \rho - \epsilon k_1 \rho^T w + \epsilon \rho^T S(w) J w + \epsilon w^T J H(\rho) w + (w + \epsilon \rho)^T d$$

Using the fact that $\|S(w)\| = \|w\|$ and $\|H(\rho)\| \leq \frac{1}{2}(1 + \|\rho\| + \|\rho\|^2)$, we obtain that

$$\dot{V}(x) \leq -\frac{k_1}{2}\|w\|^2 - \epsilon k_2 \|\rho\|^2 + \epsilon k_1 \|\rho\| \|w\| + \|w\| \|d\| + \epsilon \|\rho\| \|d\| + I(\|w\|, \|\rho\|)$$

where

$$I(\|w\|, \|\rho\|) = -\epsilon k_2 \|\rho\|^4 - \frac{k_1}{2}\|w\|^2 \left[1 - \frac{\epsilon \lambda_{\max}(J)}{k_1}(1 + 3\|\rho\| + \|\rho\|^2)\right]$$

If we design the gain $k_1$ such that

$$k_1 > 3\sqrt{\frac{c_2}{c_1}} \epsilon \lambda_{\max}(J) \max\{1, 3\|x_0\|, \|x_0\|^2\} \tag{5}$$

then we can guarantee that $I(\|w\|, \|\rho\|) \leq 0$, and we are left with the following terms:

$$\dot{V}(x) \leq -\frac{1}{2}\begin{bmatrix} \|w\| \\ \|\rho\| \end{bmatrix}^T \begin{bmatrix} k_1 & \epsilon k_1 \\ \epsilon k_1 & \epsilon k_2 \end{bmatrix} \begin{bmatrix} \|w\| \\ \|\rho\| \end{bmatrix} + \begin{bmatrix} \|w\| \\ \|\rho\| \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} \|d\| \\ \|d\| \end{bmatrix}$$

Note that the term $\sqrt{c_2/c_1}$ in Eq. (5) is included to guard against the initial overshoot of the system, which can be characterized through the quadratic bounds in Eq. (4). Also, the constant 3 is included because we took the maximum among three terms.

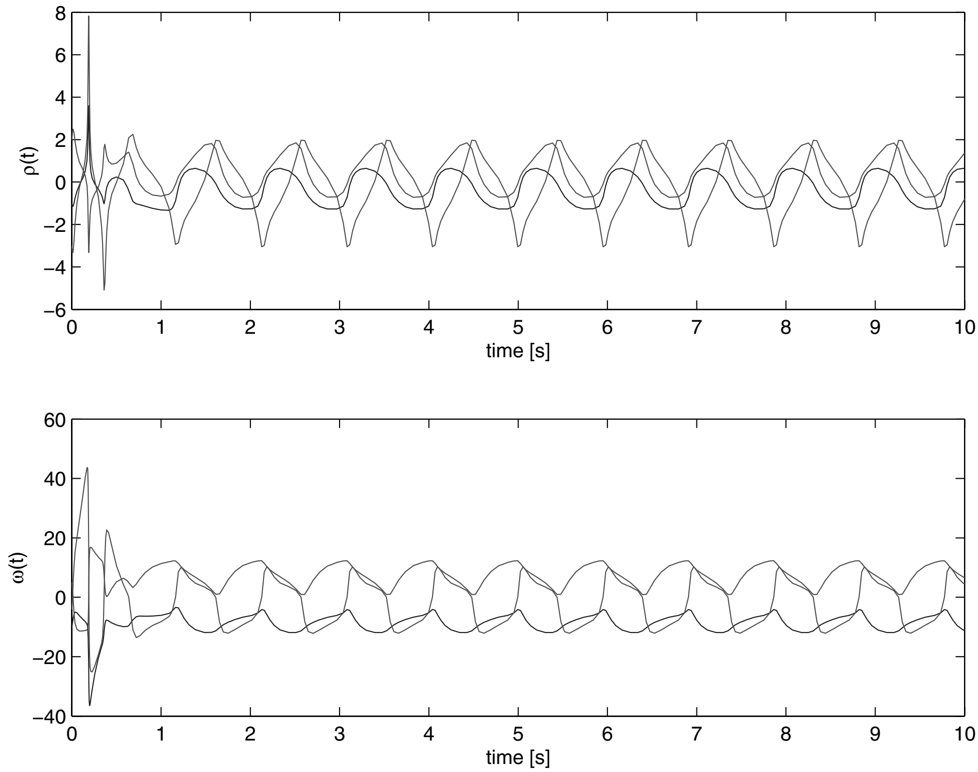


**Fig. 2 Response of the system under control law (2).**

Take any constant $c_3 > 0$, and design $k_1$ and $k_2$ such that

$$c_3\begin{bmatrix}1 & 0\\ 0 & \epsilon\end{bmatrix} \le \frac{1}{2}\begin{bmatrix}k_1 & \epsilon k_1\\ \epsilon k_1 & \epsilon k_2\end{bmatrix}$$

The latter condition can be guaranteed under the following design condition:

$$\min\{k_1, k_2\} > 2c_3 \tag{6}$$

whenever

$$\epsilon < 2\frac{(k_1 - 2c_3)(k_2 - c_3)}{k_1^2}$$

Accordingly, we can bound $\dot{V}(x)$ as follows:

$$\dot{V}(x) \le -c_3\begin{bmatrix}\|w\|\\ \|\rho\|\end{bmatrix}^T\begin{bmatrix}1 & 0\\ 0 & \epsilon\end{bmatrix}\begin{bmatrix}\|w\|\\ \|\rho\|\end{bmatrix} + \begin{bmatrix}\|w\|\\ \|\rho\|\end{bmatrix}^T\begin{bmatrix}1 & 0\\ 0 & \epsilon\end{bmatrix}\begin{bmatrix}\|d\|\\ \|d\|\end{bmatrix}$$

For any parameter $\theta \in (0, 1)$ the derivative of $V(x)$ can be bounded by a strictly negative quadratic function as

$$\dot{V}(x) \le -\theta c_3\begin{bmatrix}\|w\|\\ \|\rho\|\end{bmatrix}^T\begin{bmatrix}1 & 0\\ 0 & \epsilon\end{bmatrix}\begin{bmatrix}\|w\|\\ \|\rho\|\end{bmatrix} \le -\theta\epsilon c_3(\|w\|^2 + \|\rho\|^2) \le -\frac{\theta\epsilon c_3}{c_2}V(x) \tag{7}$$

as long as

$$\min\{\|\rho\|, \|w\|\} > \frac{\|d\|}{(1-\theta)c_3}$$

or, in terms of the full state,

$$\|x\| > \frac{\sqrt{2}\|d\|}{(1-\theta)c_3}$$

Finally, using similar arguments as in [6] (Theorem 4.18), we can show that whenever the bounds (4) and (7) hold in the restricted regions for $V(x)$ and $\dot{V}(x)$, they can be transformed into the following bound on the state $x$,

$$\|x(t)\| \le \max\left\{2\sqrt{\frac{c_2}{c_1}}e^{-\frac{\theta\epsilon c_3}{2c_2}t}\|x_0\|, \sqrt{\frac{c_2}{c_1}}\frac{2\sqrt{2}}{(1-\theta)c_3}\sup_{0\le\tau\le t}\|d(\tau)\|\right\} \tag{8}$$

and the result follows. □

Note that the gain $k_1$ is designed [Eq. (5)] depending on the initial conditions and, as such, the result is semiglobal. It is important to note that the design condition (6) on the parameters $k_1$ and $k_2$ affects the value of $c_3$, which in turn reduces the effect of the disturbance on the response of the system as seen through Eq. (8). Moreover, the convergence rate of the undisturbed system is exponential.

A very similar result can easily be derived for the case of the modified Rodrigues parametrization, in which the kinematic model and control law are given by

$$\dot{q} = G(q)w, \qquad J\dot{w} = S(w)Jw + u$$

with control input $u = -k_1 w - k_2(1 + q^T q)q$ and

$$G(q) = \frac{1}{2}\left(I - S(q) + qq^T - \frac{1 + q^T q}{2}I\right)$$

In this case, we use the upper bound

$$\|G(q)\| \le \tfrac{1}{2}\left(\tfrac{3}{2} + \|q\| + \tfrac{3}{2}\|q\|^2\right)$$

to design the control gain $k_1$ as in Eq. (5).

## IV. Example

We simulate the same scenario as in Sec. II, but using the nonlinear control law given in Eq. (2). Using the same gains as before $k_1 = k_2 = 10$, we can calculate the bounds in Eq. (4) as $c_1 = 0.4$ and

$c_2 = 10.1$. Then we pick $c_3 = 4.9$, $\epsilon = 0.01$, and $\theta = 0.01$, from which we obtain [using Eq. (5) and the condition following Eq. (7)] that the restriction values are $\Delta_x = 11.5$ and $\Delta_d = 39.4$. However, due to the fact that our result is only sufficient, the simulation in Fig. 2 indicates that the system under similar conditions of Sec. II retains stability for $\|d\| = 100 > \Delta_d$ and $\|x_0\| = 12.5 > \Delta_x$. To guarantee semiglobal input-to-state stability of the system under the latter initial conditions, a sufficiently large value for the control gains can be calculated as $k_1 = k_2 = 25$.

## V. Conclusions

We showed through an example that simple PD-type attitude control for the Cayley–Rodrigues parametrization may suffer from the lack of robustness with respect to bounded disturbances. We then proposed using a nonlinear control law, and we proved a sufficient condition under which the closed-loop system is semiglobally input-to-state stable. The result is constructive in the sense that given any restriction on the initial state and disturbance input, it is possible to design large enough control gains that can render the system input-to-state stable within the given restriction.

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